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An example concerning the L_p-stability of piecewise linear B-wavelets

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Abstract

In this paper we consider B-wavelets of order 2, i.e. piecewise linear spline prewavelets of smallest support, over nonuniform knot sequences. We discuss an example showing that for $1 < p \leq \infty$, there is no absolute L_p-stability for these B-wavelets. This means that regardless what specific scaling of the B-wavelets is chosen, the corresponding stability constants cannot be made independent of the knot sequences involved.

1 Introduction

Polynomial splines are fundamental tools in numerous branches of applied mathematics, and for spline spaces defined over a given knot sequence, the basis of choice is provided by B-splines, which possess a lot of attractive properties for numerical computations. One of these important properties of B-splines is their absolute stability. Given a B-spline basis $\{B_i\}_{i\in\mathcal{I}}$ of polynomial order d over a valid knot sequence t, a classical result by de Boor [1] states that properly normalized B-splines are stable in the sense that for each set $\{b_i\}_{i\in\mathcal{I}}$ of real coefficients it holds that

$$C_d^{-1}\left\|\mathbf{b}\right\|_p \leq \left\|\sum_{i\in\mathcal{I}} b_i \delta_i^{-\frac{1}{p}} B_i\right\|_p \leq \left\|\mathbf{b}\right\|_p. \tag{1.1}$$

Here $\|\cdot\|_p$ denotes the standard integral and discrete p-norms for $1 \leq p \leq \infty$, respectively, and the normalizing factor δ_i for each B-spline is the length of its support divided by the order d. The important point is that the positive constant C_d is dependent on the order d alone, and not in any way on the underlying knot sequence t.

Since nested knot sequences give rise to nested spline spaces, spline functions have also become a focus of attention within the theory of wavelets and multiresolution analysis,

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starting with cardinal spline wavelets on infinite equally spaced and uniformly refined knot sequences, for which Fourier transform techniques are available, see [3] and the references therein.

The study of spline wavelets on bounded intervals, for arbitrary knot sequences and nonuniform refinement began with the papers [4], [5] and [2], respectively. The construction of so-called minimally supported B-wavelets for a given spline order d and two nested knot sequences to provide a basis of the relative orthogonal complement (wavelet) space is described in detail in [6]. This means that given the coarse and fine knot sequence, there exist explicit algorithms to determine the supports of the B-wavelet functions, the so-called minimal intervals, and also to compute the corresponding wavelet functions, though only up to a normalization constant.

One open problem, however, is how to fix the normalization factor for each B-wavelet function to achieve best possible stability for the whole B-wavelet basis. We provide an example for the case of piecewise linear wavelets, i.e. polynomial order 2, that shows that for $1 < p \leq \infty$ there is no absolute stability of B-wavelets, meaning that there is no choice of normalization that provides absolute stability constants which are completely independent of the underlying knot sequences. L_p-stability estimates involving a quantity dependent on the knot sequences for $1 < p \leq \infty$ and showing absolute stability for $p = 1$ are given in [7].

2 Piecewise linear B-wavelets

The theory of B-wavelets [6] covers general cases of knot refinement, such as situations where several or no knots at all are inserted into an old knot interval, or where the multiplicity of an existing knot is increased. For our purposes, however, it is sufficient to consider what one might call the standard setting, where all knots are simple except at the interval endpoints, which we can count as double knots, and where exactly one new knot is inserted strictly between two old ones.

Our notations are as follows for the closed interval $[0, 1]$. We have a coarse knot sequence with $n - 1$ interior knots, namely

$$\tau : 0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1.$$

Strictly between each pair of coarse knots τ_{i-1} and τ_i we insert a new knot s_i at an arbitrary location, i.e.

$$\tau_{i-1} < s_i < \tau_i \text{ for each } i = 1, \ldots, n.$$

Thus we have a sequence s of new knots

$$s : 0 < s_1 < \cdots < s_n < 1.$$

The fine knot sequence $t = \tau \cup s$, when ordered appropriately, is given as

$$t : 0 = t_0 < t_1 < \cdots < t_{2n} = 1,$$

where the even numbered knots in t correspond to old knots in τ, while the odd numbered knots represent the newly inserted knots from s. To account for the boundary, we treat the interval endpoints as double knots by setting $\tau_{-1} = t_{-1} = 0$ and $\tau_{n+1} = t_{2n+1} = 1$.

For our investigations it is necessary to introduce also some notation related to the knot spacings. We set

$$d_i = t_{i+1} - t_i \text{ for } i = 0, \ldots, 2n-1, \text{ and } \delta_i = t_{i+1} - t_{i-1} \text{ for } i = 0, \ldots, 2n,$$

which means $\delta_0 = d_0 = t_1 - t_0$ and $\delta_{2n} = d_{2n-1} = t_{2n} - t_{2n-1}$ at the boundary. Thus δ_i is the distance between two consecutive old knots if i is odd, and between two consecutive new knots if i is even (and not at the boundary).

We also introduce the index sets

$$\Omega = \{1, 3, \ldots, 2n-1\} \text{ and } \Omega_0 = \{3, 5, \ldots, 2n-3\}.$$

The piecewise linear functions on the knot sequences $\tau \subset t$ form nested linear spaces $V_0 \subset V_1$ of dimensions $n+1$ and $2n+1$, respectively. The corresponding *piecewise linear B-splines* forming a basis of these spaces are simple hat functions. We denote them as φ_j and γ_i for τ and t, respectively, where with the necessary adjustments at the endpoints,

$$\varphi_j(x) = \begin{cases} (x - \tau_{j-1}) / \delta_{2j-1} & \text{if } x \in [\tau_{j-1}, \tau_j] \\ (\tau_{j+1} - x) / \delta_{2j+1} & \text{if } x \in [\tau_j, \tau_{j+1}] \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 0, \ldots, n, \tag{2.1}$$

$$\gamma_i(x) = \begin{cases} (x - t_{i-1}) / d_{i-1} & \text{if } x \in [t_{i-1}, t_i] \\ (t_{i+1} - x) / d_i & \text{if } x \in [t_i, t_{i+1}] \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } i = 0, \ldots, 2n. \tag{2.2}$$

Using for any two functions $f, g \in V_1$ the standard inner product

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt,$$

we can write

$$V_1 = V_0 \oplus W,$$

where W is the relative orthogonal complement of V_0 in V_1, and $\oplus$ denotes orthogonal summation. The dimension of W is n, so that there is a basis function ψ_k for every index $k \in \Omega$, or in other words for each newly inserted knot s_k.

Nonzero functions $\psi_k \in W$ with minimal support are called *B-wavelets*. The general theory for B-wavelets developed in [6] establishes in this special case that there are n different piecewise linear B-wavelets which form a basis of the wavelet space W. Each such B-wavelet is uniquely determined up to a constant multiple. There are two boundary B-wavelets ψ_1 and ψ_{2n-1} and $n-2$ interior B-wavelets ψ_k for $k \in \Omega_0$, which we will consider first. Each interior B-wavelet has support $[t_{k-3}, t_{k+3}]$, so that

$$\psi_k(x) = \sum_{i=k-2}^{k+2} q_i^k \gamma_i(x) \qquad \text{for } x \in [0, 1]$$

with the coefficients determined by $\psi_k \in W$, or in other words

$$\langle \psi_k, \varphi_j \rangle = 0 \qquad \text{for } j = 0, \ldots, n.$$

For the boundary wavelets ψ_1 and ψ_{2n-1} we have to make some minor modifications. Their supports are $[t_0, t_4]$ and $[t_{2n-4}, t_{2n}]$, respectively, so that

$$\psi_1(x) = \sum_{i=0}^{3} q_i^1 \gamma_i(x) \text{ and } \psi_{2n-1}(x) = \sum_{i=2n-3}^{2n} q_i^{2n-1} \gamma_i(x) \text{ for } x \in [0,1].$$

In the paper [7] the values of all B-wavelet coefficients q_i^k are given explicitly in terms of the knot locations for the standard setting described here. In the same paper estimates for the coefficients are used to derive L_p-stability estimates for these B-wavelets.

3 Stability of B-wavelets

Our aim in this paper is to establish

Theorem 3.1 *Given the B-wavelet basis $\{\psi_k\}_{k\in\Omega}$, then for $1 < p \leq \infty$, there are no sets of weights $\alpha_{k,p}$, $k \in \Omega$, such that*

$$K_1 \|\mathbf{c}\|_p \leq \left\| \sum_{k\in\Omega} c_k \alpha_{k,p} \psi_k \right\|_p \leq K_2 \|\mathbf{c}\|_p \tag{3.1}$$

holds for any wavelet coefficients $(c_1, c_3, \ldots, c_{2n-1})$ and with absolute constants $K_1 > 0$ and $K_2 > 0$, which are completely independent from the choice of knot sequences τ and s.

Due to the finite dimension of W, it is clear that stability constants K_1 and K_2 exist, as any two norms on W are equivalent. The pertinent question is how the weights could be chosen to achieve that the constants are actually independent of the dimension, the p-norm and, if possible, the choice of new knots s. We will prove the assertion by assuming that the estimate (3.1) holds with constants independent of the knot sequences. Then the following special case serves as a counterexample to this assertion.

The old knot sequence τ consists of the equally spaced points:

$$\tau_0 = 0, \tau_1 = 1/3, \tau_2 = 2/3, \tau_3 = 1.$$

We want to investigate what happens if two newly inserted points are positioned ever more closely, so we introduce the new knots as

$$s_1 = 1/3 - \varepsilon, s_2 = 1/3 + \eta, s_3 = 5/6, \text{ for } 0 < \varepsilon, \eta < 1/3,$$

in order to find out what happens if both $\varepsilon \to 0^+$ and $\eta \to 0^+$.

Thus the fine knot sequence t is

$$t_0 = 0, t_1 = 1/3 - \varepsilon, t_2 = 1/3, t_3 = 1/3 + \eta, t_4 = 2/3, t_5 = 5/6, t_6 = 1.$$

The fine interval lengths are

$$d_0 = 1/3 - \varepsilon, d_1 = \varepsilon, d_2 = \eta, d_3 = 1/3 - \eta, d_4 = 1/6, d_5 = 1/6,$$

while

$$\delta_1 = \delta_3 = \delta_5 = 1/3, \text{and } \delta_0 = 1/3 - \varepsilon, \delta_2 = \varepsilon + \eta, \delta_4 = 1/2 - \eta, \delta_6 = 1/6.$$

In this setting any wavelet

$$\psi = \sum_{i=0}^{6} q_i \gamma_i \in W$$

must be orthogonal to the coarse hat functions $\varphi_0, \cdots, \varphi_3$. This actually means that the column vector $\mathbf{q}$ of coefficients q_i must satisfy the matrix equation

$$\mathbf{Aq} = \mathbf{0}, \tag{3.2}$$

where the entries of $\mathbf{A}$ are the inner products of the coarse and fine hat functions, i.e.

$$a_{j,i} = \langle \varphi_j, \gamma_i \rangle, \quad \text{for } j = 0, \ldots, 3, \quad i = 0, \ldots, 6.$$

Direct computations using (2.1) and (2.2) yield as the only nonzero entries

$$\begin{aligned}
a_{0,0} &= -\frac{1}{2}\varepsilon^2 - \frac{1}{6}\varepsilon + \frac{1}{9}, \quad a_{0,1} = \frac{1}{6}\varepsilon + \frac{1}{18}, \quad a_{0,2} = \frac{1}{2}\varepsilon^2, \\
a_{1,0} &= \frac{1}{2}\varepsilon^2 - \frac{1}{3}\varepsilon + \frac{1}{18}, \quad a_{1,1} = -\frac{1}{6}\varepsilon + \frac{1}{9}, \\
a_{1,2} &= -\frac{1}{2}\varepsilon^2 + \frac{1}{2}\varepsilon - \frac{1}{2}\eta^2 + \frac{1}{2}\eta, \\
a_{1,3} &= -\frac{1}{6}\eta + \frac{1}{9}, \quad a_{1,4} = \frac{1}{2}\eta^2 - \frac{1}{3}\eta + \frac{1}{18}, \\
a_{2,2} &= \frac{1}{2}\eta^2, \quad a_{2,3} = \frac{1}{6}\eta + \frac{1}{18}, \\
a_{2,4} &= -\frac{1}{2}\eta^2 - \frac{1}{6}\eta + \frac{13}{72}, \\
a_{2,5} &= \frac{1}{12}, \quad a_{2,6} = \frac{1}{72}, \\
a_{3,4} &= \frac{1}{72}, \quad a_{3,5} = \frac{1}{12}, \quad a_{3,6} = \frac{5}{72}.
\end{aligned}$$

We now investigate the B-wavelets ψ_1 and ψ_3 in detail, corresponding to $s_1 = t_1$ and $s_2 = t_3$. Specializing the results from [7] then yields all necessary B-wavelet coefficients for this setting up to a scaling factor. Note, however, that it is straightforward to check that the corresponding coefficient vectors satisfy the matrix equation (3.2).

The coefficients of the boundary wavelet ψ_1 are

$$\begin{aligned}
q_0^1 &= -\frac{3}{1-3\varepsilon}, \\
q_1^1 &= 6 - \frac{9\varepsilon\eta}{\varepsilon + \eta + 6\varepsilon\eta}, \\
q_2^1 &= -\frac{1+3\eta}{\varepsilon + \eta + 6\varepsilon\eta}, \\
q_3^1 &= \frac{9\eta^2}{\varepsilon + \eta + 6\varepsilon\eta},
\end{aligned}$$

while the ones for the interior B-wavelet ψ_3 are

$$\begin{aligned} q_1^3 &= \frac{9\varepsilon^2}{\varepsilon+\eta+6\varepsilon\eta}, \\ q_2^3 &= -\frac{1+3\varepsilon}{\varepsilon+\eta+6\varepsilon\eta}, \\ q_3^3 &= 3+\frac{3(\varepsilon+\eta)}{2(\varepsilon+\eta+6\varepsilon\eta)}+\frac{9(1-2\eta)}{2(5-12\eta)}, \\ q_4^3 &= -\frac{9}{5-12\eta}, \\ q_4^3 &= \frac{3}{2\,(5-12\eta)}. \end{aligned}$$

We first provide estimates for the p-norms of these B-wavelets.

Proposition 3.2 *For small enough ε and η, it holds for $1 < p \leq \infty$ that*

$$\begin{aligned} \|\psi_1\|_p &\geq \frac{16}{45}\left(\frac{1}{2}\right)^{1/p}(\varepsilon+\eta)^{1/p-1}, \\ \|\psi_3\|_p &\geq \frac{16}{45}\left(\frac{1}{2}\right)^{1/p}(\varepsilon+\eta)^{1/p-1}. \end{aligned}$$

Proof: For all $0 < \varepsilon, \eta < 1/3$ we find that

$$\begin{aligned} \left|q_2^1\right| &\geq (\varepsilon+\eta)^{-1} \inf_{0<\varepsilon,\eta<1/3} \frac{(1+3\eta)\,(\varepsilon+\eta)}{\varepsilon+\eta+6\varepsilon\eta} \\ &= \frac{8}{9}\,(\varepsilon+\eta)^{-1} \end{aligned}$$

and, similarly,

$$\left|q_2^3\right| \geq \frac{8}{9}\,(\varepsilon+\eta)^{-1}.$$

Note that instead of 8/9 we may write $1-\sigma$ for any $\sigma > 0$ if ε and η are small enough or even 1 if $\varepsilon = \eta$.

In the process $\varepsilon, \eta \to 0^+$ all other coefficients q_i^1 and q_i^3 have finite limits. This means that for small enough $\varepsilon+\eta$

$$\begin{aligned} \|\psi_1\|_\infty &= \max\left|q_i^1\right| = \left|q_2^1\right| \geq \frac{8}{9}\,(\varepsilon+\eta)^{-1}, \\ \|\psi_3\|_\infty &= \max\left|q_i^3\right| = \left|q_2^3\right| \geq \frac{8}{9}\,(\varepsilon+\eta)^{-1}. \end{aligned}$$

The absolute stability of piecewise linear B-splines (1.1) yields with $C_2 \geq 5/2$ (see [1]) and $\delta_2 = \varepsilon+\eta$

$$\|\psi_1\|_p = \left\|\sum_{i=0}^{3} q_i^1\gamma_i\right\|_p \geq \frac{2}{5}\left(\frac{1}{2}\right)^{1/p}\left\|\left(q_0^1\delta_0^{1/p}, q_1^1\delta_1^{1/p}, q_2^1\delta_2^{1/p}, q_3^1\delta_3^{1/p}\right)\right\|_p$$

$$\geq \frac{2}{5}\left(\frac{1}{2}\right)^{1/p} \left|q_2^1\right| \delta_2^{1/p} \geq \frac{16}{45}\left(\frac{1}{2}\right)^{1/p} (\varepsilon+\eta)^{1/p-1}.$$

Analogously we get

$$\|\psi_3\|_p = \left\|\sum_{i=1}^{5} q_i^3 \gamma_i\right\|_p \geq \frac{2}{5}\left(\frac{1}{2}\right)^{1/p} \left|q_2^3\right| \delta_2^{1/p} \geq \frac{16}{45}\left(\frac{1}{2}\right)^{1/p} (\varepsilon+\eta)^{1/p-1}$$

to complete the proof. □

Proof of Theorem 3.1: Let us now assume that with some scaling factor B-wavelets are absolutely stable in p-norm for $1 < p \leq \infty$, i.e. there exist weights $\alpha_{k,p}$ so that the inequalities (3.1) hold with constants independent of the specific choice of knot sequences. Choosing in the current setting all coefficients equal to zero except for $c_1 = 1$, the stability inequality (3.1) yields

$$\|\alpha_{1,p}\psi_1\|_p \leq K_2$$

or in other words, using Proposition 3.2

$$|\alpha_{1,p}| \leq \frac{K_2}{\|\psi_1\|_p} \leq \frac{45}{16} 2^{1/p} K_2 (\varepsilon+\eta)^{1-1/p} \tag{3.3}$$

and by a similar argument

$$|\alpha_{3,p}| \leq \frac{K_2}{\|\psi_3\|_p} \leq \frac{45}{16} 2^{1/p} K_2 (\varepsilon+\eta)^{1-1/p}. \tag{3.4}$$

On the other hand, the stability estimate (3.1) yields for arbitrary c_1 and c_3, while setting all other c_k to zero, that

$$\|c_1\alpha_{1,p}\psi_1 + c_3\alpha_{3,p}\psi_3\|_p \geq K_1 \left(|c_1|^p + |c_3|^p\right)^{1/p} \geq K_1 \max\left(|c_1|, |c_3|\right).$$

Let us choose specifically

$$c_1 = \alpha_{3,p} \text{ and } c_3 = -\alpha_{1,p},$$

which results in

$$|\alpha_{1,p}\alpha_{3,p}| \, \|\psi_1 - \psi_3\|_p \geq K_1 \max\left(|\alpha_{1,p}|, |\alpha_{3,p}|\right),$$

leading with (3.3) and (3.4) to

$$(\varepsilon+\eta)^{1-1/p} \|\psi_1 - \psi_3\|_p \geq \left(\frac{1}{2}\right)^{1/p} \frac{16K_1}{45K_2}. \tag{3.5}$$

On the other hand we derive from the absolute stability of linear B-splines (1.1) that

$$\begin{aligned}
&\|\psi_1 - \psi_3\|_p \\
&= \left\|q_0^1\gamma_0 + \left(q_1^1 - q_1^3\right)\gamma_1 + \left(q_2^1 - q_2^3\right)\gamma_2 + \left(q_3^1 - q_3^3\right)\gamma_3 - q_4^3\gamma_4 - q_5^3\gamma_5\right\|_p \\
&\leq \left\|\left(q_0^1\delta_0^{1/p}, \left(q_1^1 - q_1^3\right)\delta_1^{1/p}, \left(q_2^1 - q_2^3\right)\delta_2^{1/p}, \left(q_3^1 - q_3^3\right)\delta_3^{1/p}, q_4^3\delta_4^{1/p}, q_5^3\delta_5^{1/p}\right)\right\|_p
\end{aligned}$$

$$\leq \; 6\max\Big(\left|q_0^1\right|\delta_0^{1/p}, \left|q_1^1 - q_1^3\right|\delta_1^{1/p}, \left|q_2^1 - q_2^3\right|\delta_2^{1/p}, \left|q_3^1 - q_3^3\right|\delta_3^{1/p}, \left|q_4^3\right|\delta_4^{1/p}, \left|q_5^3\right|\delta_5^{1/p}\Big).$$

All the terms

$$\left|q_0^1\right|\delta_0^{1/p}, \left|q_1^1 - q_1^3\right|\delta_1^{1/p}, \left|q_3^1 - q_3^3\right|\delta_3^{1/p}, \left|q_4^3\right|\delta_4^{1/p}, \left|q_5^3\right|\delta_5^{1/p}$$

are in fact bounded from above for $\varepsilon + \eta \to 0^+$, so that the expression

$$(\varepsilon+\eta)^{1-1/p}\left|q_0^1\right|\delta_0^{1/p}$$

and the other such terms tend to zero.

Since $\left|q_2^1 - q_2^3\right| = 3\left|\varepsilon - \eta\right| / (\varepsilon + \eta + 6\varepsilon\eta)$ we obtain for the only remaining term

$$(\varepsilon+\eta)^{1-1/p}\left|q_2^1 - q_2^3\right|\delta_2^{1/p} = \frac{3\left|\varepsilon-\eta\right|}{1+6\varepsilon\eta/\left(\varepsilon+\eta\right)} \leq 3\left|\varepsilon - \eta\right|,$$

which goes to zero as well for $\varepsilon + \eta \to 0^+$. As a consequence

$$\lim_{\varepsilon+\eta\to 0^+} (\varepsilon+\eta)^{1-1/p}\left\|\psi_1 - \psi_3\right\|_p = 0,$$

which contradicts (3.5). □

Remark 3.3 *Although we have chosen an example with one boundary and one interior B-wavelet, let us remark that the lack of absolute stability is in no way due to a boundary effect. A completely analogous reasoning is possible if one chooses knot sequences with more interior knots and studies the behaviour for two interior B-wavelets once two new knots coalesce. Similarly just two boundary B-wavelets could be used on an even shorter knot sequence, where there are no interior B-wavelets at all.*

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